Mathematics: Content Knowledge (0061)

Test Prep Kit Available
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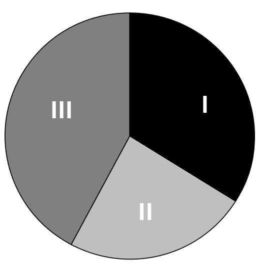
Test at a Glance

Test Name	**Mathematics: Content Knowledge**
Test Code	**0061**
Time	**2 hours**
Number of Questions	**50**
Format	**Multiple-choice questions, graphing calculator required**



	Content Categories	Approximate Number of Questions	Approximate Percentage of Examination
I.	Arithmetic and Basic Algebra	6-8	
	Geometry	4-6	
	Trigonometry	2-4	
	Analytic Geometry	2-4	
		17	34%
II.	Functions and Their Graphs	5-7	
	Calculus	5-7	
		12	24%
III.	Probability and Statistics	3-5	
	Discrete Mathematics	3-5	
	Linear Algebra	3-5	
	Computer Science	2-4	
	Mathematical Reasoning and Modeling	5-7	
		21	42%

About this test

The Content Knowledge test in Mathematics is designed to assess the mathematical knowledge and competencies necessary for a beginning teacher of secondary school mathematics. Designed to conform to the curriculum, evaluation, and professional standards of the National Council of Teachers of Mathematics, the test focuses on problem solving, communication, reasoning, and mathematical connections.

The 50 multiple-choice questions require the ability to understand and work with mathematical concepts, to reason mathematically, to integrate knowledge of different areas of mathematics, and to develop mathematical models of real-life situations.

Graphing calculators without QWERTY (typewriter) keyboards are required for this test. Some questions will require the use of a calculator. The minimum capabilities required of the calculator are described in the section on graphing calculators below. Because many test questions may be solved in more than one way, decide first how to solve each problem; then decide whether to use a calculator. On the test day, examinees should bring a calculator they are comfortable using.

Selected notations, formulas, and definitions are printed in the test book and are also listed on pages 20-22.

Graphing Calculators

You will be expected to bring to the examination a graphing calculator that can

(1) produce the graph of a function within an arbitrary viewing window
(2) find the zeros of a function
(3) compute the derivative of a function numerically
(4) compute definite integrals numerically

These capabilities may be either built into the calculator or programmed into the calculator prior to the examination. Calculator memories will *not* be cleared. Computers, calculators with QWERTY (typewriter) keyboards, and electronic writing pads are not allowed.

Unacceptable machines include the following:
Powerbooks and portable computers
Pocket organizers (Wizard, etc.)
Electronic writing pads or pen input devices (Newton, etc.)
Palm top computers with QWERTY keyboards (HP-95, TI-92, etc.)

Mathematics Content Descriptions — Basic

Representative descriptions of the topics covered in the basic content categories for the Content Knowledge and the Proofs, Models, and Problems tests follow. Because the assessments were designed to measure the ability to integrate knowledge of mathematics, answering any question may involve more than one competency and may involve competencies from more than one content area.

Arithmetic and Basic Algebra

- Understand the structure of the natural, integer, rational, real, and complex number systems; have the ability to perform the basic operations ($+$, $-$, $\times$, and $\div$) on numbers in these systems; identify properties (e.g., closure, commutativity, associativity, distributivity) of the basic operations
- Given newly defined operations on a number system, determine whether the closure, commutative, associative, or distributive properties hold
- Demonstrate an understanding of the properties of counting numbers (e.g., prime or composite, even or odd, factors, multiples)
- Solve ratio, proportion, percent, and average (including arithmetic mean and weighted average) problems
- Work with algebraic expressions, formulas, and equations
- Solve and graph systems of equations and inequalities, including those involving absolute value
- Use the results of the binomial theorem
- Present geometric interpretations of algebraic principles

Geometry

- Solve problems using relationships of parts of geometric figures (e.g., medians of triangles, inscribed angles in circles) and among geometric figures (e.g., congruence, similarity), 2-dimensional and 3-dimensional
- Describe relationships among sets of special quadrilaterals, such as the square, rectangle, parallelogram, rhombus, and trapezoid
- Solve problems using the properties of triangles, quadrilaterals, polygons, circles, parallel and perpendicular lines
- Apply the Pythagorean theorem to solve problems
- Compute perimeter, area/surface area, and volume of 2-dimensional and 3-dimensional figures

- Solve problems involving reflections, rotations, and translations of geometric figures in the plane
- Perform geometric constructions using straight-edge and compass and prove that the constructions yield the desired result

Trigonometry

- Define and use the six basic trigonometric relations using degree or radian measure of angles; know their graphs and be able to identify their period, amplitude, phase displacement or shift, and asymptotes
- Know and use the trigonometric functions of special angles (e.g., $\pi/6$, $2\pi/3$, $9\pi/4$, $-\pi/3$)
- Apply the law of sines and the law of cosines
- Apply the formulas for the trigonometric functions of $x/2$, $2x$, $x + y$, and $x - y$ in terms of the trigonometric functions of x and y
- Prove identities using the basic trigonometric identities
- Solve trigonometric equations and inequalities
- Convert between the rectangular and polar coordinate systems
- Find the trigonometric form of complex numbers and apply DeMoivre's theorem

Analytic Geometry

- Determine the equations of lines and planes
- Make calculations in 2-space or 3-space (e.g., distance between two points, coordinates of a midpoint of a line segment, distance between a point and a plane)
- Translate between the geometric definition of a conic section and its equation

Functions and Their Graphs

- Understand and be able to work with functions and their graphs, including functions given as mappings
- Given an equation, graph it; given a graph, determine an equation for it
- Determine properties of a function, such as domain, range, intercepts, symmetries, intervals of increase or decrease, discontinuities, asymptotes
- Use the properties of algebraic, trigonometric, logarithmic, and exponential functions to solve problems (e.g., finding composite functions and inverse functions)
- Find the inverse of a one-to-one function in simple cases and know why one-to-one functions have inverses

Calculus

- Know what it means for a function to have a limit at a point; calculate limits of functions or determine that the limit does not exist; solve problems using the properties of limits
- Know when and how to use L'Hospital's rule
- Show that a particular function is continuous; understand the difference between continuity and differentiability
- Relate the derivative of a function to a limit and to the slope of a curve
- Use standard differentiation and integration techniques
- Make numerical approximations of derivatives and integrals
- Analyze the behavior of a function (e.g., find relative maxima and minima, concavity); solve problems involving related rates, rates of change, approximation of roots of a function; solve applied minima-maxima problems
- Understand and be able to use the Mean Value Theorem and the Fundamental Theorem of Calculus
- Demonstrate an intuitive understanding of the process of integration
- Evaluate improper integrals
- Calculate the area of regions in the plane; calculate the volume of solids formed by rotating plane figures about a line
- Determine the limits of sequences and simple infinite series
- Use standard tests to show convergence (either conditional or absolute) or divergence of series (e.g., comparison, ratio)

Probability and Statistics

- Organize data into a suitable form (e.g., construct a histogram and use it in the calculation of probabilities)
- Solve discrete and joint probability problems; know when events are independent and how to calculate the probability of independent events
- Solve problems using the binomial distribution and be able to determine when the use of the binomial distribution is appropriate
- Find and know the appropriate uses of common measures of central tendency (population mean, sample mean, median, mode) and dispersion (range, population standard deviation, sample standard deviation, population variance, sample variance)
- Model problems using mathematical expectation of a random variable (e.g., fair coins, expected winnings, expected profit)
- Solve problems using the normal, uniform, and chi-square distributions
- Recognize a valid test to determine whether to accept or reject a given null hypothesis (H_0)

Discrete Mathematics

- Know the basic terminology of symbolic logic; use truth tables to verify statements; apply laws of Algebra of Propositions to evaluate equivalence of complex logical expressions (e.g., De Morgan's laws)
- Perform elementary operations on sets
- Solve basic problems involving permutations and combinations
- Use the Euclidean Algorithm to find the greatest common divisor of two numbers
- Work with numbers expressed in bases other than base ten
- Find values of functions defined recursively; "translate" between recursive and closed form expressions for a function
- Determine if a binary relation on a set is reflexive, symmetric, antisymmetric, transitive, or an equivalence relation
- Solve simple linear programming problems

Linear Algebra

- Scalar multiply, add, subtract, and multiply vectors and matrices
- Find inverses of matrices; understand and use the properties of inverses of matrices
- Determine and apply the matrix representation of a linear transformation
- Use matrix techniques to solve systems of linear equations

Computer Science

- Demonstrate an understanding of the roles of the hardware and software components of a computer system
- Know basic computer terminology
- Develop and debug computer algorithms (written in pseudocode)

Mathematical Reasoning and Modeling

- Develop a mathematical model; determine if one model will describe two different situations
- Determine appropriate problem-solving strategies and consider alternatives. Strategies might include conjectures, counterexamples, inductive reasoning, deductive reasoning (mathematical induction, proof by contradiction, direct proof, other types of proof) and deciding which tools are appropriate (e.g., discussion with others, mental math, pencil and paper, calculator, computer, trees and graphs, fingers)
- Recognize the reasonableness of results
- Estimate answers; determine the accuracy of an estimate by analyzing the effects of roundoff and truncation errors introduced in the course of solving a problem
- Demonstrate an understanding of the different levels of mathematical impossibility, such as: "I lack the mathematical skills to do it"; "No one has been able to do it as yet" (e.g., prove Goldbach's conjecture); "No one will ever be able to do it" (e.g., trisect a general angle with straight edge and compass)
- Use the axiomatic method

Selected Notations, Formulas, and Definitions (as provided in the test book)

NOTATIONS

(a,b)	$\{ x : a < x < b \}$
$[a,b)$	$\{ x : a \leq x < b \}$
$(a,b]$	$\{ x : a < x \leq b \}$
$[a,b]$	$\{ x : a \leq x \leq b \}$
$\gcd(m,n)$	greatest common divisor of two integers m and n
$\text{lcm}(m,n)$	least common multiple of two integers m and n
$[x]$	greatest integer m such that $m \leq x$
$m \equiv k \pmod{n}$	m and k are congruent modulo n (m and k have the same remainder when divided by n, or equivalently, $m - k$ is a multiple of n)
f^{-1}	inverse of an invertible function f; (not to be read as $1/f$)
$\lim_{x \to a^+} f(x)$	right-hand limit of $f(x)$; limit (if it exists) of $f(x)$ as x approaches a from the right
$\lim_{x \to a^-} f(x)$	left-hand limit of $f(x)$; limit (if it exists) of $f(x)$ as x approaches a from the left
$p \to q$	statement p implies statement q; if p, then q
$p \leftrightarrow q$	statements p and q are logically equivalent; p if and only if q
$\sim p$	negation of the statement p; not p
$p \wedge q$	conjunction of statements p and q; p and q
$p \vee q$	disjunction of statements p and q; p or q
$p \veebar q$	exclusive or for statements p and q; p or q but not both
$B \leftarrow C$	(in a computer algorithm) value of the variable B is replaced by the current value of the variable C
$\varnothing$	the empty set
$x \in S$	x is an element of set S
$S \subset T$	set S is a proper subset of set T
$S \subseteq T$	either set S is a proper subset of set T or $S = T$
$\overline{S}$	complement of set S; the set of all elements not in S that are in some specified universal set
$T \setminus S$	relative complement of set S in set T, i.e., the set of all elements of T that are not elements of S
$S \cup T$	union of sets S and T
$S \cap T$	intersection of sets S and T

FORMULAS

Sum
$\sin(x + y) = \sin x \cos y + \cos x \sin y$
$\cos(x + y) = \cos x \cos y - \sin x \sin y$

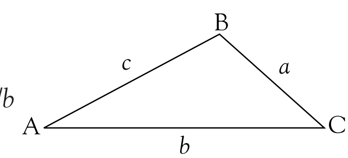
Half-Angle (sign depends on the quadrant of $\theta/2$)

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}; \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

Range of inverse trigonometric functions

$\sin^{-1} x \ [-\pi/2, \pi/2]$ $\quad$ $\cos^{-1} x \ [0, \pi]$ $\quad$ $\tan^{-1} x \ (-\pi/2, \pi/2)$

Law of Sines $\quad$ $\sin A/\sin B = a/b$



Law of Cosines $\quad$ $c^2 = a^2 + b^2 - 2ab(\cos C)$

DeMoivre's Theorem $\quad$ $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Coordinate Transformation
rectangular (x,y) to polar (r,θ) $\quad$ $r^2 = x^2 + y^2$; $\tan\theta = y/x$ if $x \neq 0$
polar (r,θ) to rectangular (x,y) $\quad$ $x = r\cos\theta$; $y = r\sin\theta$

Distance from point (x_1,y_1) to line $\quad$ $Ax + By + C = 0$ $\quad$ $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

Volume

Sphere: radius r $\quad$ $V = \frac{4}{3}\pi r^3$

Right circular cone: height h, base of radius r $\quad$ $V = \frac{1}{3}\pi r^2 h$

Right circular cylinder: height h, base of radius r $\quad$ $V = \pi r^2 h$

Pyramid: height h, base of area B $\quad$ $V = \frac{1}{3}Bh$

Right prism: height h, base of area B $\quad$ $V = Bh$

Surface Area

Sphere: radius r $\quad$ $A = 4\pi r^2$

Lateral surface area of right circular cone: radius r, slant height s $\quad$ $A = \pi rs$

Differentiation $\quad$ $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ $\quad$ $(f[g(x)])' = f'[g(x)]g'(x)$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \text{ if } g(x) \neq 0$$

Integration by Parts $\quad$ $\int u\,dv = uv - \int v\,du$

Selected Notations, Formulas, and Definitions (as provided in the test book)

DEFINITIONS

Linear Algebra

A vector $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ if there exist real numbers $a_1, a_2, a_3, \ldots, a_n$ such that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \ldots + a_n\mathbf{v}_n.$$

The linear span of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$.

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ are linearly independent if
$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \ldots + a_n\mathbf{v}_n = \mathbf{0}$ (the zero vector)
implies that $a_1 = a_2 = a_3 = \ldots = a_n = 0$.

A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$ forms a basis for a subspace W of $\mathbf{R}^n$
if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$ are linearly independent and their linear span is equal to W.

The dimension of a subspace W of $\mathbf{R}^n$ is the number of vectors in a basis for W.

Discrete Mathematics

A relation $\Re$ on a set S is:

reflexive if $x \Re x$ for all $x \in S$
symmetric if $x \Re y \Rightarrow y \Re x$ for all $x, y \in S$
transitive if $(x \Re y$ and $y \Re z) \Rightarrow x \Re z$ for all $x, y, z \in S$
antisymmetric if $(x \Re y$ and $y \Re x) \Rightarrow x = y$ for all $x, y \in S$

An equivalence relation is a reflexive, symmetric, and transitive relation.

The sample questions that follow illustrate the kinds of questions in the test. They are not, however, representative of the entire scope of the test in either content or difficulty. Answers with explanations follow the questions.

Directions: Each of the questions or statements below is followed by four suggested answers or completions. Select the one that is best in each case.

Arithmetic and Basic Algebra

1. Which of the following properties is satisfied by the integers under subtraction?

   I. Closure
   II. Associativity
   III. Commutativity

   (A) I only
   (B) II only
   (C) I and II only
   (D) I, II, and III

2. What is the sum of the coefficients in the binomial expansion of $(x + y)^{18}$?

   (A) 38
   (B) 153
   (C) 2^{18}
   (D) 2^{20}

3. If $y < z$ and $|x - y| - |x - z| = 0$, which of the following could be true?

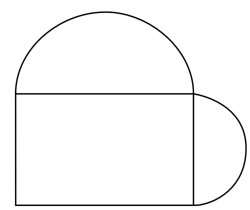
   I. $x < y$
   II. $y < x < z$
   III. $z < x$

   (A) I only
   (B) II only
   (C) I and III only
   (D) II and III only

4. Which of the following is equal to i^{59}?

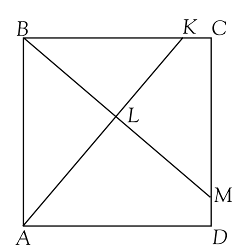
   (A) i
   (B) -1
   (C) $-i$
   (D) 1

Geometry



5. In the figure above, the adjacent sides of a rectangle form diameters of two semicircular regions. The rectangular region has an area of 81. If the ratio of the areas of the semicircular regions is 16 to 1, what is the perimeter of the rectangle?

   (A) 27
   (B) 45
   (C) 54
   (D) 65



6. Points K and M are chosen on the sides BC and CD, respectively, of the square $ABCD$, as shown in the figure above. If $\frac{KC}{BK} = \frac{1}{5}$ and $\frac{MD}{CM} = \frac{1}{5},$ what is the measure, in degrees, of $\angle KLM$?

   (A) 25
   (B) 50
   (C) 75
   (D) 90

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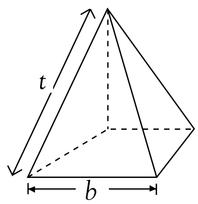


7. The pyramid above has a square base with sides of length b and lateral edges of length t. If the altitude of the pyramid is h, what is the value of t in terms of b and h ?

   (A) $\sqrt{\frac{b^2}{2}+\frac{h^2}{2}}$

   (B) $\sqrt{b^2+\frac{h^2}{2}}$

   (C) $\sqrt{\frac{b^2}{2}+h^2}$

   (D) $\sqrt{b^2+h^2}$

Trigonometry

8. On which of the following intervals is the function $f(x) = \cos(2x + \pi)$ strictly increasing?

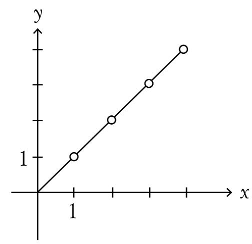
   (A) $(-\pi/4, \pi/4)$
   (B) $(0, \pi/2)$
   (C) $(\pi/4, 3\pi/4)$
   (D) $(3\pi/4, 5\pi/4)$

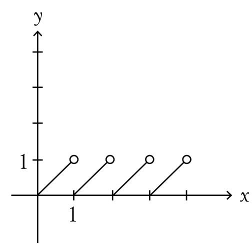
Functions and Their Graphs

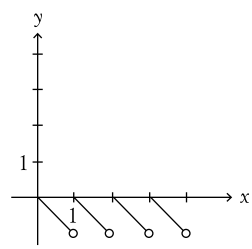
9. What is the range of the function $y = \dfrac{x^2 - 1}{x^2 + 1}$?

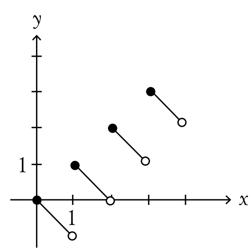
   (A) $-1 < y < 1$
   (B) $-1 \le y < 1$
   (C) All real numbers except -1
   (D) All real numbers

10. For any real number x, let $[x]$ denote the greatest integer that is less than or equal to x. Which of the following is the graph of the function $g(x) = 2[x] - x$ for $0 \le x < 4$?

   (A)
   

   (B)
   

   (C)
   

   (D)
   

11. At how many points in the xy-plane do the graphs of $y = 0.25x^4 + 0.4x^3 - 1.2x^2 - 0.75x + 2$ and $y = 0.5x + 0.01$ intersect?

    (A) One
    (B) Two
    (C) Three
    (D) Four

Probability and Statistics

12. If a student takes a test consisting of 20 true-false questions and randomly guesses at all of the answers, what is the probability that all 20 guesses will be correct?

    (A) 0
    (B) $\left(\frac{1}{2}\right)^{20}$
    (C) $\frac{1}{2(20)}$
    (D) $\frac{1}{2}$

Analytic Geometry

13. If (i) the graph of the function $f(x)$ is the line with slope 3 and y-intercept 1, and

    (ii) the graph of the function $g(x)$ is the semicircle in the upper half plane with center at the origin and radius 2,

    what is the domain of $g(f(x))$?

    (A) $[0, 2]$
    (B) $[-1, 1/3]$
    (C) $[-2, 2]$
    (D) $(-\infty, \infty)$

Calculus

14. What is the area, to 2 decimal places, of the region in the first quadrant of the xy-plane that is bounded above by the curve $y = \cos x$ and below by the line $y = x$?

    (A) 0.36
    (B) 0.40
    (C) 0.60
    (D) 0.84

15. Let $g(x) = \sin^3 x + \cos^3 x$, and let ℓ be the line tangent to the curve $y = g(x)$ at the point on the curve $y = g(x)$ where $x = 0.93$. Which of the following is closest to the y-intercept of ℓ?

    (A) -1.5
    (B) 0.5
    (C) 1
    (D) 1.25

16. $\lim_{x \to \infty} \frac{x^9}{15x^9 - 1,000x}$ is

    (A) $-\frac{1}{1,000}$
    (B) 0
    (C) $\frac{1}{15}$
    (D) nonexistent

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Mathematics: Content Knowledge (0061)

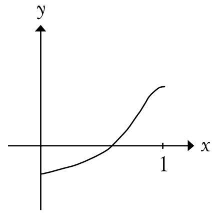
17. For all x in (0, 1)

    (i) f is continuous at x and

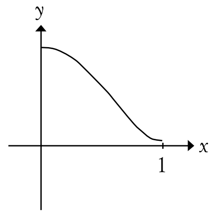
    (ii) $\int_0^x f(t)\, dt$ is a strictly increasing function of x.

    Which of the following CANNOT be the graph of f ?

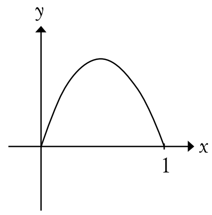
(A)



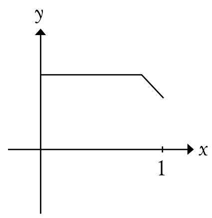
(B)



(C)



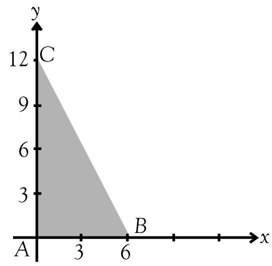
(D)



18. If $f(x) = x^5 - 7x^3 + 6x^2 - 3x + 17$, for how many values of x is the line tangent to the graph of f at the point $(x, f(x))$ horizontal?

    (A) One
    (B) Two
    (C) Three
    (D) Four

Discrete Mathematics



19. How many points with integral coordinates lie inside or on the boundary of ΔABC above?

    (A) 36
    (B) 42
    (C) 49
    (D) 65

Linear Algebra

20. The matrix representation A for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ is $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. If $\mathbf{v} = (-1, 1)$, then $T(\mathbf{v}) =$

    (A) (1, 1)
    (B) (1, 2)
    (C) (2, 1)
    (D) (2, 2)

Computer Science

21. Step 1: $A = 80, N = 1$
    Step 2: Print the value of A
    Step 3: A ← ___?___
    Step 4: $N \leftarrow N + 1$
    Step 5: If $N \leq 7$ go to Step 2
    Step 6: End program

    Which of the following is needed in Step 3 so that the output generated by this program will be the sequence 80, 40, 20, 10, 5, 2, 1?

    (A) $\sqrt{A}$
    (B) $[A]$
    (C) $\dfrac{A}{2}$
    (D) $\left[\dfrac{A}{2}\right]$

Mathematical Reasoning and Modeling

22. Which of the following is equal to the number of 13-element subsets of a set of 50 elements?

    I. The number of ways a bag of 13 candies can be assembled from an assortment of 50 different pieces of candy.
    II. The coefficient of x^{13} in the expansion of $(x + 1)^{50}$.
    III. The number of ways in which 13 prizes ranked from first to thirteenth can be awarded among a group of 50 people if no one can receive more than one prize.

    (A) I and II only
    (B) I and III only
    (C) II and III only
    (D) I, II, and III

23. In order to estimate the population of snails in a certain woodland, a biologist captured and marked 84 snails that were then released back into the woodland. Fifteen days later the biologist captured 90 snails from the woodland, 12 of which bore the markings of the previously captured snails.

    If all of the marked snails were still active in the woodland when the second group of snails were captured, what should the biologist estimate the snail population to be based on the probabilities suggested by this experiment?

    (A) 630
    (B) 1,010
    (C) 1,040
    (D) 1,080

Answers

1. When you are asked if a certain property is always true, it is useful to ask yourself, "Can I find a case when the property is not satisfied?" If so, you have found a counterexample. On the other hand, if after trying several cases you cannot find a counterexample, you can begin to suspect that the property is always satisfied. However, to be sure that the property is always satisfied you must show that it holds in every case.

I. Closure — Is the difference between two integers always an integer? Look at several cases, such as $3 - 2 = 1$, $7 - 8 = -1$, $7 - 30 = -23$. After trying several cases we see that we have not found a counterexample. So we begin to suspect that the integers are closed under subtraction. In fact, when the integers are "built" from the natural numbers it is closure under subtraction that is achieved.

II. Associativity — Is $(a - b) - c = a - (b - c)$ for all integers a, b, and c? Look at a specific case. For example, let $a = 1, b = 2$, and $c = 3$.

$$(a - b) - c = (1 - 2) - 3 = -1 - 3 = -4$$

$$a - (b - c) = 1 - (2 - 3) = 1 - (-1) = 1 + 1 = 2$$

Notice that these are not equal. Thus the integers are not associative under subtraction.

III. Commutativity — Is $a - b = b - a$ for all integers a and b? Look at a specific case. For example, if $a = 1$ and $b = 2$, then $a - b = 1 - 2 = -1$ while $b - a = 2 - 1 = 1$.

We conclude that of the three properties, only I is satisfied by the integers under subtraction, and thus the correct answer is A.

2. One approach to this problem is to write out the binomial expansion

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \ldots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

To get the sum of the desired coefficients, set $x = 1, y = 1, n = 18$. The equation becomes

$$(1 + 1)^{18} = \binom{18}{0} + \binom{18}{1} + \ldots + \binom{18}{17} + \binom{18}{18},$$

and we see that the sum of the coefficients is 2^{18}. Hence the correct answer is C.

Another approach is to look for a pattern in the coefficients of $(x + y)^n$ as n increases. Recall that the coefficients of the binomial expansion of $(x + y)^n$ can be read from the n^{th} row of Pascal's triangle, and hence their sum can be obtained using Pascal's triangle and look for a pattern.

n	Pascal's Triangle	Sum of Coefficients of $(x + y)^n$
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32

Continuing the pattern, it seems reasonable to conclude that the sum of the coefficients of $(x + y)^{18}$ is 2^{18} and the correct answer is C. (Note: This is not a proof, but a conjecture based on a pattern.)

3. The two absolute value expressions would be easier to handle if they were separated. Separating them yields $|x - y| = |x - z|$. Recall that $|x - y|$ represents the distance between x and y on the number line. If the equation above is thought of in this way, it translates to, "What point x on the number line is the same distance from the point y as it is from the point z on the number line?" Draw the number line, with $y < z$,

and move along it until you find a point x that is the same distance from y as it is from z. Doing this you can see that $y < x < z$; and thus the correct answer is B.

This problem can also be approached algebraically.

$$|x - y| - |x - z| = 0$$

$$|x - y| = |x - z|$$

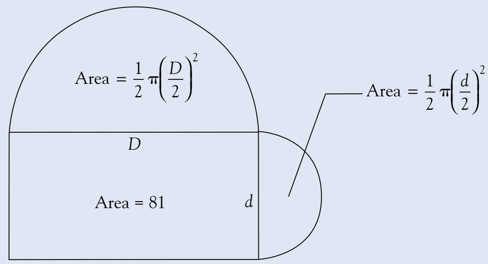
so either $x - y = x - z$, and thus $y = z$, or else $x - y = -(x - z)$, and thus $x = \frac{y + z}{2}$ (i.e., x is the average of y and z). Since we are given that $y < z$, it follows that x is the average of y and z, and therefore $y < x < z$ and the correct answer is B.

4. The problem asks us to raise i to the 59th power. One approach is to raise i to successively higher powers and see if a pattern emerges.

$$\begin{aligned} i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \\ i^6 &= -1 \\ &\vdots \end{aligned}$$

Notice that the powers of i repeat in cycles of 4: $i, -1, -i, 1, i, -1, -i, 1, \ldots$. Therefore, $i^{4p} = 1$ for every positive integer p. Since $59 = (4)(14) + 3$, it follows that $i^{59} = i^{(4)(14)+3} = i^{(4)(14)}i^3 = (1)(-i) = -i$, and hence the correct answer is C.

5. In solving problems of this kind it is often useful to label the figure with the information given in the problem. Let D be the diameter of the larger semicircle and d the diameter of the smaller semicircle.



Since the ratio of the area of the larger semicircular region to the smaller semicircular region is 16 to 1, it follows that

$\frac{1}{2}\pi\left(\frac{D}{2}\right)^2 = 16\left(\frac{1}{2}\pi\left(\frac{d}{2}\right)^2\right)$, or $D^2 = 16d^2$, or $D = 4d$.

The only other piece of information we were given is that the area of the rectangular region, or Dd, is 81. Since $Dd = 81$ and $D = 4d$, it follows that $81 = Dd = 4d^2$, or $d = \frac{9}{2}$, and hence $D = 18$. The perimeter of the rectangle is therefore $2(D + d) = 2\left(18 + \frac{9}{2}\right) = 45$, and the correct answer is B.

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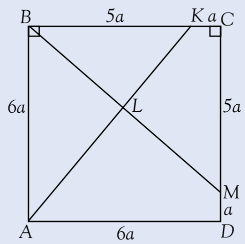
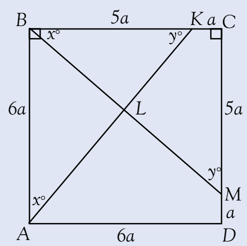
(Answers continued)

6. Let the length of KC equal a. Since $\frac{KC}{BK} = \frac{1}{5}$, the length of BK is $5a$. Since $ABCD$ is a square and $\frac{MD}{CM} = \frac{1}{5}$, the figure can be labeled as follows:



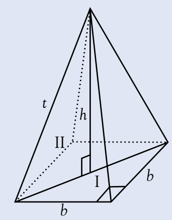
From the figure you can see that ΔABK is congruent to ΔBCM. Thus, we can further label the equal angles of the two congruent right triangles.



Since $x + y = 90$, it follows that the measure of $\angle BLC = 90°$, and therefore the measure of $\angle KLM = 90°$ and the correct answer is D.

Another approach is to notice that a square is symmetric and that the two line segments AK and BM are constructed using the same "specifications" (i.e., the side of the square is broken into segments whose lengths have the ratio 1:5). Because of these properties of the figure, you can then rotate the square about its center by 90° counterclockwise and line segment BM will be rotated to line segment AK. Since this means that a rotation of 90° brings BM to AK, the angle at which the two line segments intersect is 90° and the correct answer is D.

7. All the triangular faces of the pyramid are congruent and the bottom face of the pyramid is a square. Therefore, the altitude of the pyramid intersects the square at its center. This implies that the altitude bisects the diagonals of the base. Right triangles can be formed as shown in the figure below:

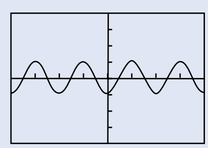


By the Pythagorean theorem applied to triangle I, the diagonal of the square base is $\sqrt{2}\,b$, and therefore the length of the leg of triangle II that lies in the square base of the pyramid is $\frac{\sqrt{2}}{2}b$. Applying the Pythagorean theorem to triangle II gives

$$\left(\frac{\sqrt{2}}{2}b\right)^2 + h^2 = \frac{b^2}{2} + h^2 = t^2. \text{ Hence } t = \sqrt{\frac{b^2}{2} + h^2}$$

and the correct answer is C.

8. When determining the behavior of a function, it is usually useful to begin by graphing the function. With the viewing widow $-2\pi \le x \le 2\pi$ and $-4 \le y \le 4$, the calculator gives this graph:



The function is increasing on the interval $\left(0, \frac{\pi}{2}\right)$, and thus the correct answer is B.

9. The question asks you to find all possible values of y. From the options, you can see that you can answer the question by answering these two questions:

(1) Can y ever be greater than 1?
(2) Can y ever equal -1?

(1) In answering questions like (1), it is sometimes helpful to work backwards. In this case, begin with the inequality $y < 1$ and see where that takes you, If $y < 1$, then $\frac{x^2 - 1}{x^2 + 1} < 1$. Since $x^2 + 1 > 0$, you can multiply both sides of this inequality by $x^2 + 1$, obtaining $x^2 - 1 < x^2 + 1$. Subtracting x^2 from both sides gives the true statement $-1 < 1$.

The fact that you have arrived at a true statement does not show that $y < 1$, but it does provide you with a suggested approach for demonstrating that $y < 1$ — begin with $-1 < 1$ and work backward:

$$-1 < 1$$
$$x^2 - 1 < x^2 + 1$$
$$\frac{x^2 - 1}{x^2 + 1} < 1$$
$$y < 1$$

It follows that $y < 1$ for all real x, and therefore the correct answer cannot be C or D.

(2) To see if y can ever equal -1, you can try to solve the equation $y = -1$:

$$\frac{x^2 - 1}{x^2 + 1} = -1$$
$$x^2 - 1 = -(x^2 - 1) = -x^2 + 1$$
$$2x^2 = 0$$
$$x = 0$$

You can now check that when $x = 0$, indeed $y = -1$. It follows that the correct answer cannot be A. Therefore, the correct answer is B.

10. There are several ways to approach this problem. One way is to compute some values of $g(x)$. Since $[x] = x$ for integer values of x, you can construct the following table of values:

x	$g(x)$
0	0
1	1
2	2
3	3

Since the graph in option D is the only graph consistent with this table, the correct answer must be D.

Another approach to the problem is to find equations that define $g(x)$ piecewise. Since

$$[x] = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \\ 3 & \text{if } 3 \le x < 4 \end{cases}$$

it follows that

$$g(x) = \begin{cases} 2 \cdot 0 - x = -x & \text{if } 0 \le x < 1 \\ 2 \cdot 1 - x = 2 - x & \text{if } 1 \le x < 2 \\ 2 \cdot 2 - x = 4 - x & \text{if } 2 \le x < 3 \\ 2 \cdot 3 - x = 6 - x & \text{if } 3 \le x < 4 \end{cases}$$

Each piece of this graph has a negative slope, which excludes options A and B, and $g(1) = 1$, which excludes option C. Hence, the correct answer is D.

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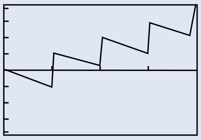
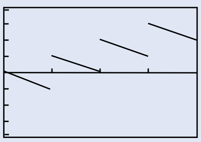
(Answers continued)

A third approach is to graph the function on a graphing calculator. With the viewing window $0 \le x \le 4$ and $-4 \le y \le 4$, the calculator gives this graph:

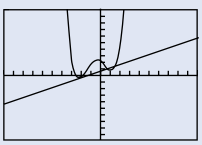


This graph does not look exactly like any of the options because the calculator is trying to make the function continuous, and it is not. But if you ignore the vertical line segments at $x = 1, 2,$ and 3, you can see that the calculator's graph looks like the one in option D. Alternatively, you can graph the function with the calculator in Dot mode instead of Connected mode.

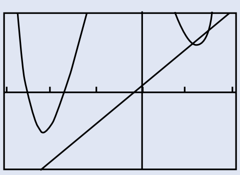


Again, you can see that the correct answer is D.

11. To find the number of points at which the graphs of two functions intersect, it is helpful to graph the two functions on a graphing calculator. For the function above, with the viewing window $-10 \le x \le 10$ and $-10 \le y \le 10$, the calculator gives this graph:



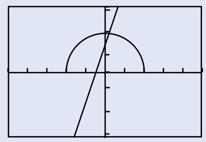
It is unclear from this view how many intersection points there are, but it is clear that the only intersection points occur in the interval $-3 \le x \le 2$. With the viewing window $-3 \le x \le 2$ and $-1 \le y \le 1$, the calculator gives this graph:



From this view it is clear that the graphs have exactly two points of intersection. The correct answer is B.

12. The probability that the student guesses any one answer correctly is 1/2, and, since the student is randomly guessing, the guesses are independent events. Thus, the probability of guessing all 20 answers correctly is $\left(\frac{1}{2}\right)^{20}$, and the correct answer is B.

13. The domain of $g(f(x))$ is the set of all x for which $g(f(x))$ is defined. You can begin by graphing $f(x)$ and $g(x)$ on a graphing calculator. With the viewing window $-4.7 \le x \le 4.7$ and $-3.1 \le y \le 3.1$, the calculator gives this graph:



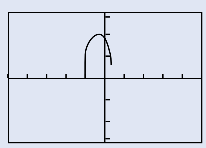
From the graph you can see that $g(x)$ is defined for all x such that $-2 \le x \le 2$. Alternatively, $g(x)$ is given by the equation $g(x) = \sqrt{4 - x^2}$, and thus its domain is the set of all x for which $4 - x^2 \ge 0$, or $x^2 \le 4$, or $-2 \le x \le 2$.

Either way, it follows that $g(f(x))$ is defined for all x such that $-2 \le f(x) \le 2$. Since $f(x)$ is given by the equation $f(x) = 3x + 1$, the domain of $g(f(x))$ is the set of all x satisfying the double inequality

$$-2 \le 3x + 1 \le 2$$
$$-3 \le 3x \le 1$$
$$-1 \le x \le \frac{1}{3}$$

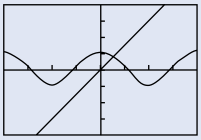
It follows that the domain of $g(f(x))$ is the closed interval $\left[-1, \frac{1}{3}\right]$; thus the correct answer is B.

Alternatively, you can let $Y_1 = 3x + 1$, $Y_2 = \sqrt{4 - x^2}$, and $Y_3 = Y_2(Y_1(x))$, and graph Y_3. With the viewing window $-4.7 \le x \le 4.7$ and $-3.1 \le y \le 3.1$, the calculator gives this graph:

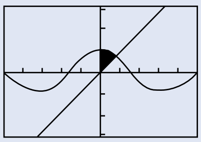


B is the only option consistent with this graph; therefore the correct answer is B.

14. You can begin by graphing the functions $y = \cos x$ and $y = x$ on a graphing calculator. With the viewing window $-2\pi \le x \le 2\pi$ and $-4 \le y \le 4$, the calculator gives this graph:

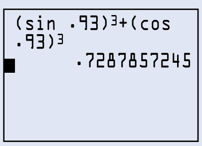


The area to be computed is the area of the shaded region in the graph below.

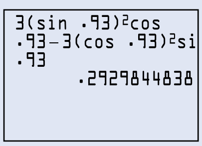


This area is given by the integral $\int_0^b (\cos x - x)\,dx$, where b is the x-coordinate of the point of intersection of the cosine curve and the line. According to the calculator, the x-coordinate of the point of intersection, to three decimals, is 0.739. Therefore, the area is $\int_0^{0.739} (\cos x - x)\,dx \approx 0.40$. The correct answer is B.

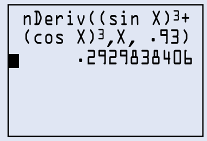
15. The equation of the tangent line is $y - b = m(x - a)$, where $a = 0.93$, $b = g(0.93)$ and the slope $m = g'(0.93)$. The calculator can be used to compute $g(0.93)$:



You can compute $g'(0.93)$ by computing the derivative $g'(x) = 3\sin^2 x \cos x - 3\cos^2 x \sin x$ and evaluating $g'(0.93)$ on the calculator:

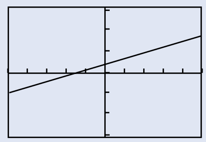


Alternatively, you can compute $g'(0.93)$ directly on the calculator:



Either way, you can see that the equation of the tangent line, with the coefficients rounded to two decimal places, is $y - 0.73 = 0.29(x - 0.93)$ or $y = 0.29(x - 0.93) + 0.73$. The y-intercept of this line is the point where $x = 0$. This is the point $y = (0.29)(-0.93) + 0.73 \approx 0.46$.

Alternatively, you can graph the tangent line on a graphing calculator. With the viewing window $-4.7 \le x \le 4.7$ and $-3.1 \le y \le 3.1$, the calculator gives this graph:



You can see from the graph that the y-intercept is approximately 0.5. Therefore, the correct answer is B.

16. There are several ways to approach this problem One way is to observe that, as $x \to \infty$, x^9 becomes much larger than x. Therefore, for large values of x, $15x^9 - 1{,}000x \approx 15x^9$, and hence

$$\frac{x^9}{15x^9 - 1{,}000x} \approx \frac{x^9}{15x^9} = \frac{1}{15}.$$ Thus,

$$\lim_{x\to\infty} \frac{x^9}{15x^9 - 1{,}000x} = \frac{1}{15}$$ and the correct answer is C.

Another way to approach the problem is to divide the numerator and the denominator by the highest power of x that appears in the fraction, in this case, x^9:

$$\lim_{x\to\infty} \frac{x^9}{15x^9 - 1{,}000x} = \lim_{x\to\infty} \frac{x^9}{\frac{15x^9 - 1{,}000x}{x^9}}$$

$$= \lim_{x\to\infty} \frac{1}{15 - \frac{1{,}000}{x^8}} = \frac{1}{15}$$

since $\lim_{x\to\infty} \frac{1{,}000}{x^8} = 0$.

(continued on page 34)

(Answers continued)

A third way to approach the problem is first to algebraically simplify the fraction, and then to apply L'Hospital's Rule. L'Hospital's Rule is applicable because the numerator and the denominator both tend to ∞ as x tends to ∞.

$$\lim_{x\to\infty} \frac{x^9}{15x^9 - 1,000x} = \lim_{x\to\infty} \frac{x^8}{15x^8 - 1,000} =$$

$$\lim_{x\to\infty} \frac{8x^7}{8 \cdot 15x^7} = \frac{1}{15}$$

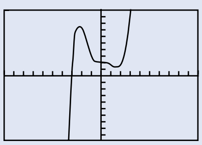
17. Note the following two facts:

(i) All of the graphs in the options are continuous.

(ii) For a continuous function f, $\int_0^x f(t)\,dt$ is decreasing at x if $f(x)$ is negative and increasing at x if $f(x)$ is positive.

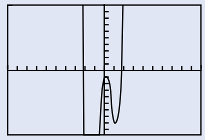
The functions in choices B, C, and D are positive for all $x \in (0, 1)$ and therefore $\int_0^x f(t)\,dt$ is strictly increasing on $(0, 1)$. The function in choice A is negative on part of the interval thus $\int_0^x f(t)\,dt$ is not strictly increasing for all $x \in (0, 1)$ and the correct answer is A.

18. You can begin by graphing $f(x)$ on a graphing calculator. With the viewing window $-10 \le x \le 10$ and $-10 \le y \le 10$, the graph does not show enough of the curve, but with the viewing window $-10 \le x \le 0$ and $-100 \le y \le 100$, the calculator gives the following graph:



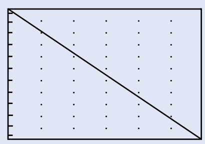
From the graph it seems that the graph of f has a horizontal tangent at approximately $x = -2$ and another at approximately $x = 2$, but it is not clear whether the graph of f has any horizontal tangents elsewhere.

Since the horizontal tangents of the graph of f correspond to zeros (x-intercepts) of f', the next thing to try is to graph the derivative $f'(x) = 5x^4 - 21x^2 + 12x - 3$ on the calculator. With the viewing window $-10 \le x \le 10$ and $-10 \le y \le 10$, the calculator gives the following graph:



Since f' is a fourth-degree polynomial, this graph shows all of its zeros. Since f' has only two zeros, it follows that the graph of f has a horizontal tangent at only two points. Thus, the correct answr is B.

19. You can begin by reproducing the above graph on a graphing calculator with the grid on and a tickmark at each integer point. The line CB has slope -2 and y-intercept 12, and hence its equation is $y = -2x + 12$. You can graph the function $y = -2x + 12$ on the calculator with the viewing window $0 \le x \le 6$ and $0 \le y \le 12$, with the grid on, and with tickmarks at each integer point; the calculator gives this graph:



The line $y = -2x + 12$ intersects the y-axis at 12; between 12 and 0 on the y-axis there are 13 points with integral coordinates. Along the line $x = 1$, there are 13 points with integral coordinates, but 2 of them are above the line $y = -2x + 12$ and hence are not in the region. Thus, along the line $x = 1$ there are 11 points with integral coordinates in the region. Similarly, along the line $x = 2$, there are 9 points with integral coordinates in the region, and so on. It follows that the total number of points with integral coordinates in the region is $13 + 11 + 9 + 7 + 5 + 3 + 1 = 49$. Thus, the correct answer is C.

20. To say that A is the matrix representation of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ means that for any $\mathbf{v} \in \mathbf{R}^2$, $T(\mathbf{v})$ is the matrix product $A\mathbf{v}$, where $\mathbf{v}$ is written as a column matrix. It follows that in this problem,

$T(\mathbf{v}) = A\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore, the correct answer is A.

21. In analyzing programs like this, it is often helpful to begin by working through the program step by step. Here is an analysis of this program:

In Step 1, value of A is 80 and the value of N is 1.
In Step 2, the value of A, 80, is printed.
In Step 3, the value of A is changed to an unknown value.
In Step 4, the value of N is changed to 2.
In Step 5, since $N = 2 \leq 7$, the program goes back to step 2.
In the second pass through Step 2, the program prints the value of A, which is now the unknown value. Since you want the program to print 40 at this step, it follows that in the first pass through Step 3 the value of A must have been changed to 40.

A similar analysis shows that at subsequent passes through Step 3, the value of A must be changed to 20, then to 10, then to 5, then to 2, and finally to 1. Until the fifth pass, the value of A is replaced with the value of $\frac{A}{2}$, but on the fifth pass, 5 is replaced with 2 rather than 2.5. In other words, A is replaced with the greatest integer less than $\frac{A}{2}$; that is, A is replaced with $\left[\frac{A}{2}\right]$. Therefore, the correct answer is D.

22. In working this problem it is important to keep in mind the difference between the number of *combinations* of n objects taken r at a time, ${}_nC_r$, and the number of *permutations* of n objects taken r at a time, ${}_nP_r$. ${}_nC_r = \frac{n!}{r!(n-r)!}$ is the number of ways of choosing r objects from a set of n objects when the order in which the objects are chosen does not matter; the same collection of r objects chosen in a different order is regarded as the same combination.

${}_nP_r = \frac{n!}{(n-r)!}$ is the number of ways of choosing r objects from a set of n objects when the order in which the objects are chosen does matter; the same collection of r objects chosen in a different order is regarded as a different permutation.

The number of 13-element subsets of a set of 50 elements is equal to ${}_{50}C_{13}$ because the order in which the elements are chosen does not matter. Two subsets are equal if and only if they have the same elements, regardless of the order in which the elements were chosen. Therefore, you can answer this question by determining which of I, II, and III equal ${}_{50}C_{13}$.

I. Two bags of candy are considered the same if they contain the same candies; the order in which the candies were selected does not matter. Therefore, the number of ways a bag of 13 candies can be assembled from a set of 50 different candies is ${}_{50}C_{13}$.

II. According to the Binomial Theorem in algebra,

$$(x+1)^{50} = \binom{50}{50}x^{50} + \binom{50}{49}x^{49} + \ldots + \binom{50}{0}x^{0}$$

where the binomial coefficient $\binom{50}{r} = {}_{50}C_r$. It follows that the coefficient of x^{13} is $\binom{50}{13} = {}_{50}C_{13}$.

III. The order in which the prizes are awarded does matter, because some prizes are ranked higher than others. Therefore, the number of different ways the prizes can be awarded is ${}_{50}P_{13}$.

It follows that I and II only equal ${}_{50}C_{13}$, and therefore the correct answer is A.

23. Given the conditions of the experiment it is reasonable to assume that the 90 snails captured by the biologist 15 days after the markings were made were a random sample of all the snails.

Thus, about $\frac{12}{90}$, or $\frac{2}{15}$, of the population had been marked. Thus, the original 84 snails marked represented approximately $\frac{2}{15}$ of the entire population and the biologist should estimate the snail population to be $84 \times \frac{15}{2}$, or 630. The correct answer is A.